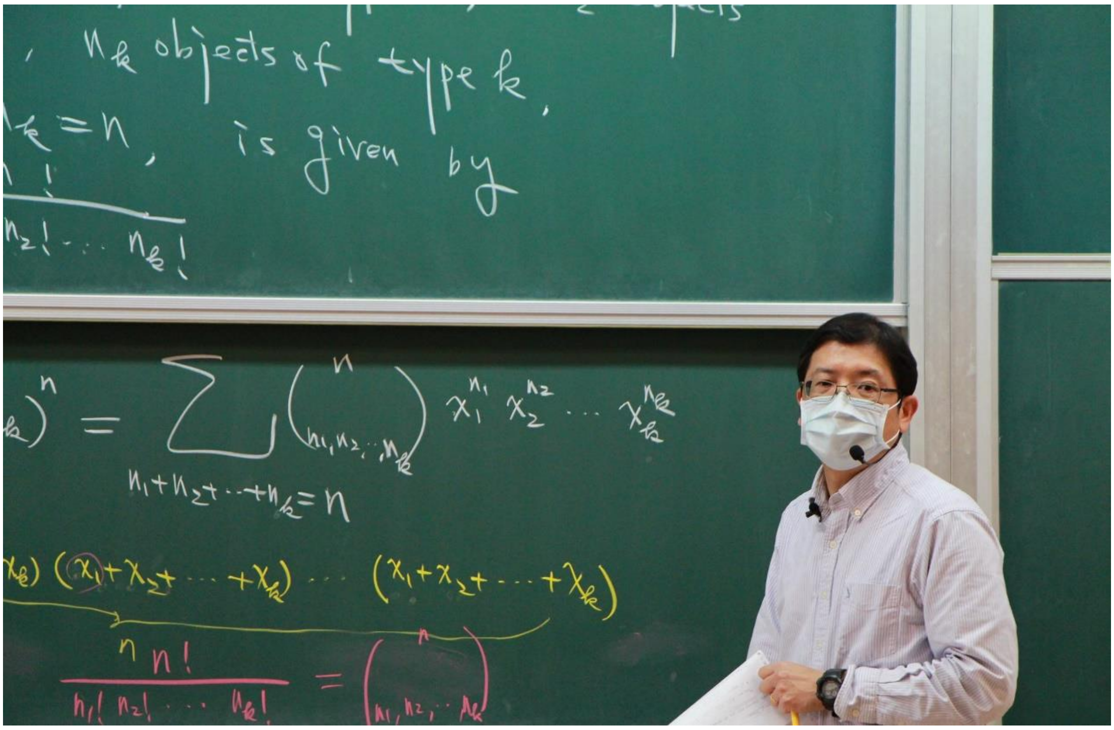
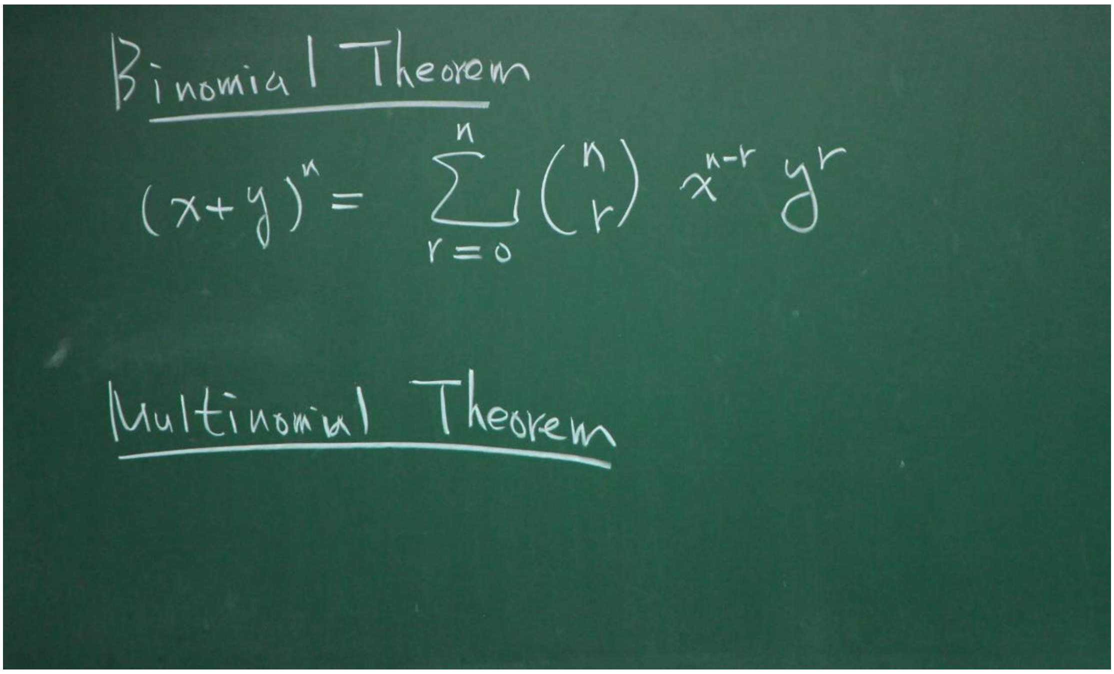
【10920 趙啟超教授離散數學 / 第 10 堂版書】





Multinomial Theorem

Example How many ways can we permute the word "MISSISSIPPI"?

There are 4 I's, 4 S's, 2 P's, 1 M.

$$\# \text{ of ways} = \frac{11!}{4!\,4!\,2!\,1!}.$$

Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Multinomial Theorem

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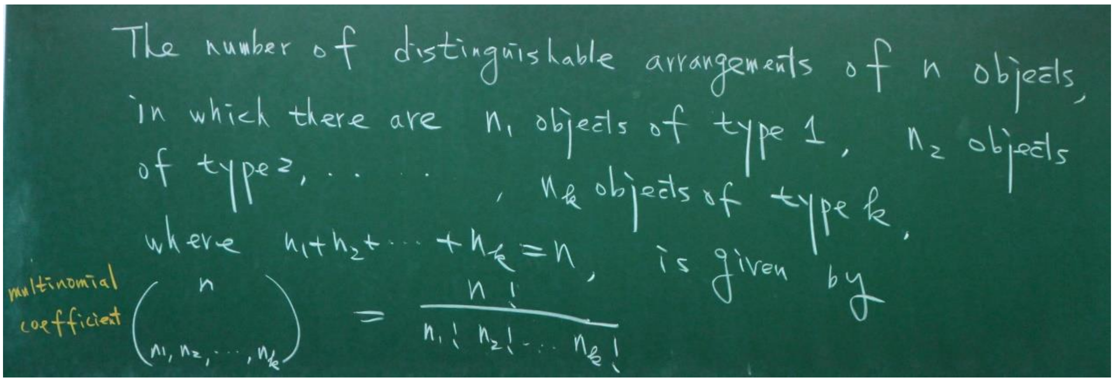
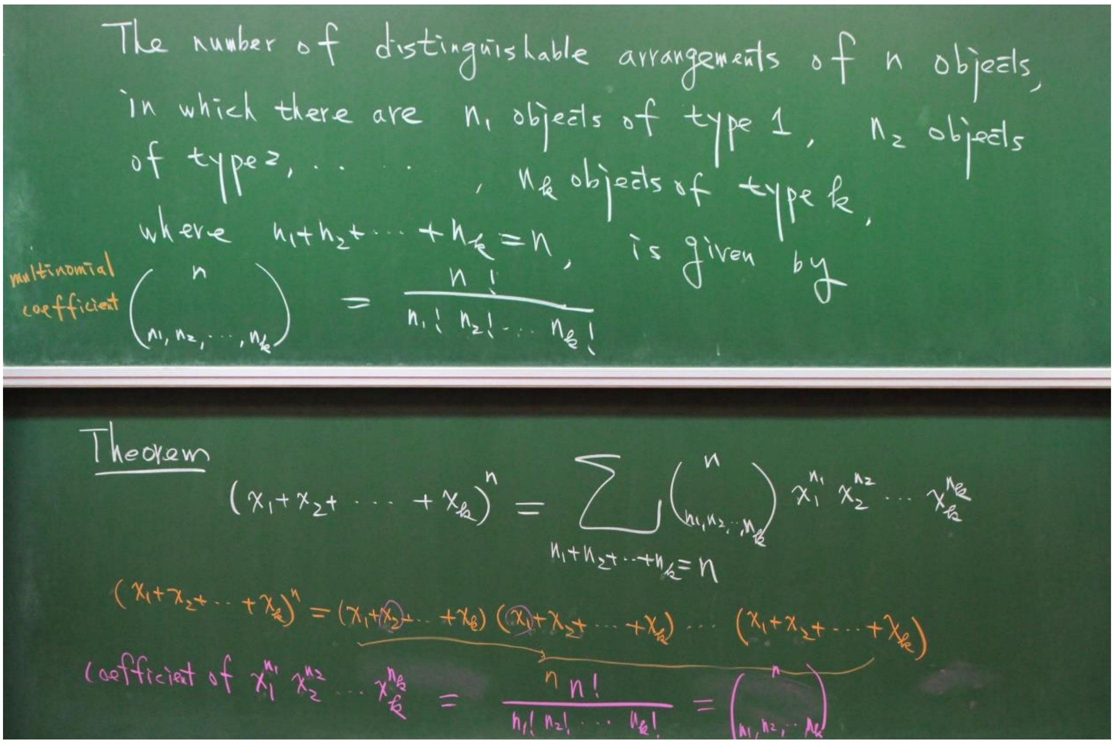

The number of distinguishable arrangements of n objects, in which there are n_1 objects of type 1, n_2 objects of type 2, . . . , n_k objects of type k, where $n_1+n_2+\cdots+n_k=n$, is given by
multinomial coefficient
$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$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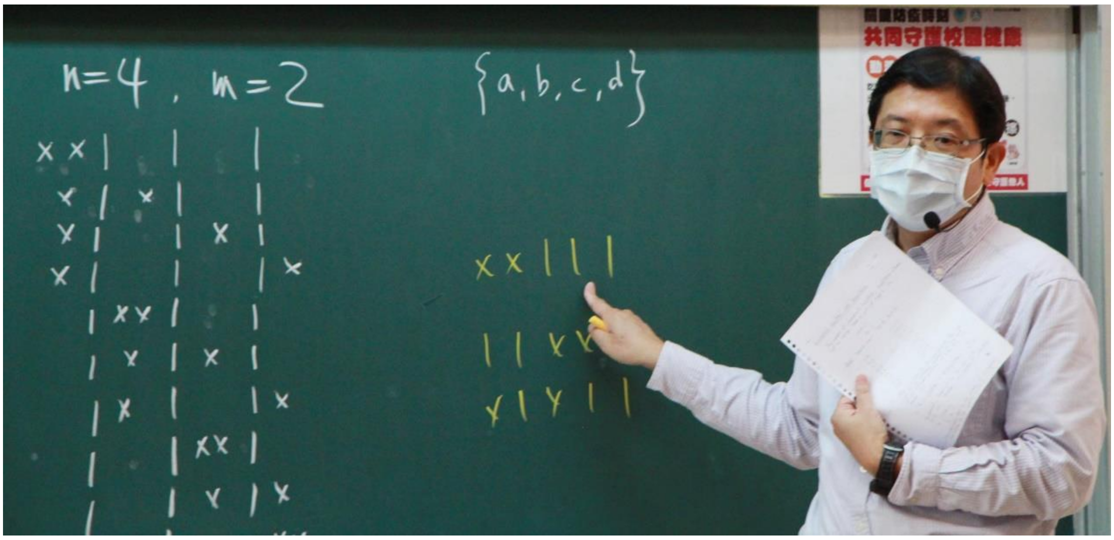
Theorem
$(x_1+x_2+\cdots+x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$
$(x_1+x_2+\cdots+x_k)^n = (x_1+x_2+\cdots+x_k)(x_1+x_2+\cdots+x_k)\cdots(x_1+x_2+\cdots+x_k)$
coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}$


$n=4,\ m=2$
$\{a, b, c, d\}$
x x | | |
x | x | |
x | | x |
x | | | x
| x x | |
| x | x |
| x | | x
| | x x |
| | x | x
x x | | |
| | x x
x | x | |

Proof Special case: $n=4$, $m=2$ $\{a,b,c,d\}$

a a	xx|||
a b	x|x||
a c	x||x|

Unordered Selections with Repetition

The number of unordered selections, repetition allowed, of m things from a set of size n is

$$\binom{n+m-1}{m}.$$

Proof Special case: $n=4$, $m=2$ $\{a,b,c,d\}$

a a	xx|||
a b	x|x||
a c	x||x|
a d	x|||x
b b	|xx||
b c	|x|x|
b d	|x||x
c c	||xx|
c d	||x|x
d d	|||xx

xx|||

||xx|

x|x||

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Proof Special case: $n=4$, $m=2$ $\quad \{a, b, c, d\}$

a a	× × | | |
a b	× | × | |
a c	× | | × |
a d	× | | | ×
b b	| × × | |
b c	| × | × |
b d	| × | | ×
c c	| | × × |
c d	| | × | ×
d d	| | | × ×

× × \| \| \|

\| \| × × \|

× \| × \| \|

This is a one-to-one correspondence from the set of all different selections to the set of all different patterns of two ×'s and three |'s.

$\therefore$ # of selections = # of patterns = $\binom{5}{2}$

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In general, there will m x's and $(n-1)$ |'s.
Therefore, the number of selections is $\binom{n+m-1}{m}$.

This number is also equal to the number of nonnegative integer solutions to the equation

$$x_1+x_2+\cdots+x_n=m.$$

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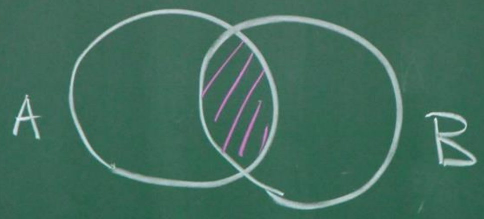
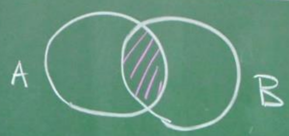
Principle of Inclusion and Exclusion



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Principle of Inclusion and Exclusion



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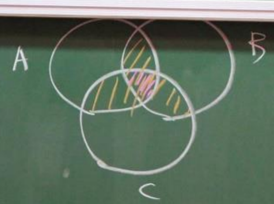
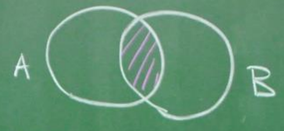


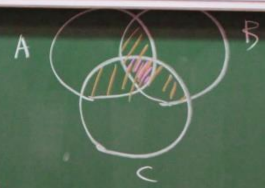
$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \\ &= \alpha_1 - \alpha_2 + \alpha_3 \end{aligned}$$

where $\alpha_1 = |A| + |B| + |C|$, $\alpha_2 = |A \cap B| + |B \cap C| + |C \cap A|$, $\alpha_3 = |A \cap B \cap C|$

Principle of Inclusion and Exclusion



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
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b	c	x	x	
b	d	x		x
c	c		xx	
c	d		x	x
d	d			xx

| x x |

x | x | |

Theorem If $A_1, A_2, \cdots, A_n$ are finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \alpha_1 - \alpha_2 + \alpha_3 - \cdots + (-1)^{n-1} \alpha_n$$

where α_i is the sum of the cardinalities of the intersections of the sets taken i at a time, $1 \le i \le n$.